

# Integral formulas for a Dirichlet series

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january 16, 2013

## Abstract

We present an integral representation formula for a Dirichlet series whose coefficients are the values of the Liouville's arithmetic function.

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# 1 Introduction

Let  $\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$  be a Dirichlet series such that :

- its analytic continuation is a meromorphic function with only one pole at  $s = 1$
- there is a functional equation looking like :

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \varphi(s) \sum_{n=1}^{\infty} \frac{b(n)}{n^{1-s}}$$

This gives a sequence  $(b(n))$  allowing us to write a pseudo-cotangent or a pseudo-tangent function similar to a cotangent or a tangent function :

$$\sum_{n=0}^{\infty} \frac{b(n)}{z^2 + (2n+1)^2 \pi^2}$$

We prefer to choose a tangent function because the cotangent function has a singularity at the origin, hence some trouble to get a power series.

It may be possible to deduce from the functional equation an integral formula for the starting Dirichlet series. Now we can hope to find a sequence  $(c(n))$  allowing us to get an extension of this integral formula by a modification of the pseudo-tangent such as :

$$\sum_{n=1}^{\infty} c(n) \frac{1}{e^{z/n} + 1}$$

An easy example is the Riemann's  $\zeta$  function itself, the three sequences are :

$$(a(n)) = (1, 1, 1, 1, \dots) \quad (b(n)) = (1, 1, 1, 1, \dots) \quad (c(n)) = (1, 0, 0, 0, \dots)$$

Another simple example is the Dirichlet series  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$  :

$$(a(n)) = (1, -1, 1, -1, \dots) \quad (b(n)) = (1, 0, 1, 0, \dots) \quad (c(n)) = (1, 0, 0, 0, \dots)$$

This is a general program. Here we take a particular case : a Dirichlet series equivalent to the Dirichlet series whose coefficients are the values of the Liouville arithmetic function. We obtain a representation integral formula.

## 2 Some functions associated with the Riemann's $\zeta$ function

### 2.1 The functions $\zeta$ , $\zeta_a$ , $\zeta_{imp}$ .

The following Dirichlet functions are well known :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \Re(s) > 1 \tag{1}$$

$$\zeta_a(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad \Re(s) > 1 \tag{2}$$

$$\zeta_{imp}(s) = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^s} \quad \Re(s) > 1 \tag{3}$$

The links with the  $\zeta$  function are easy :

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \zeta_a(s) \quad (4)$$

$$\zeta(s) = \frac{1}{1 - 2^{-s}} \zeta_{imp}(s) \quad (5)$$

Cf, for example [6] .

## 2.2 The functions $\zeta_\lambda$ , $\zeta_\mu$ , $\zeta_\alpha$ .

Let  $\lambda$  be the Liouville's arithmetic function :  $\lambda(1) = 1$ ; for a prime  $p$ ,  $\lambda(p) = -1$ ; for all  $a$  et  $b$ ,  $\lambda(ab) = \lambda(a)\lambda(b)$ .

Let  $\zeta_\lambda$  be the corresponding Dirichlet function :

$$\zeta_\lambda(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \quad \Re(s) > 1 \quad (6)$$

$$\zeta_\lambda(s) = \frac{\zeta(2s)}{\zeta(s)} \quad (7)$$

$\zeta_\lambda$  is a meromorphic function on  $\mathbb{C}$  .

Let  $\mu$  be the Möbius arithmetic fonction :

$$\zeta_\mu(s) = \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad \Re(s) > 1 \quad (8)$$

$\zeta_\lambda$  and  $\zeta_\mu$  are also well known, cf [6] .

The singular points for  $\zeta_\mu$  are the zeros of  $\zeta$ . But  $\zeta_\lambda$  has no singular points outside the domain  $0 \leq \Re(s) \leq 1$ . In that domain, its singular points are the zeros of the  $\zeta$  function, except for  $s = 1$  .

Let  $\zeta_\alpha$  be :

$$\zeta_\alpha(s) = \frac{\zeta_a(2s)}{\zeta_a(s)} \quad (9)$$

With (4) we get :

$$\zeta_\lambda(s) = \frac{1 - 2^{1-s}}{1 - 2^{1-2s}} \zeta_\alpha(s) \quad (10)$$

We do not need the arithmetic function  $\alpha$  such that :

$$\zeta_\alpha(s) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}$$

## 2.3 The function $\zeta_\beta$ .

Let  $\zeta_\beta$  be :

$$\zeta_\beta(s) = \frac{\zeta_{imp}(2s-1)}{\zeta_{imp}(s)} \quad (11)$$

With (5) we have :

$$\zeta_\beta(s) = \frac{(1 - 2^{1-2s})\zeta(2s-1)}{(1 - 2^{-s})\zeta(s)} \quad (12)$$

$\zeta_\beta$  is the generating function of an arithmetic function  $\beta$  :

$$\zeta_\beta(s) = \sum_{n=1}^{\infty} \frac{\beta(n)}{n^s} \quad \Re(s) > 1 \quad (13)$$

According to a theorem of Newman, cf [3], the following series is convergent and its value is :

$$\sum_{n=0}^{\infty} \frac{\mu(2n+1)}{2n+1} = 0 \quad (14)$$

Let  $(2n+1)$  be an odd number. There is a unique decomposition in a factor without square and a square :

$$\begin{cases} 2n+1 = kh^2 \\ \beta(2n+1) = \mu(k)h \\ |\beta(2n+1)| = h \end{cases} \quad (15)$$

We have the estimate :

$$-1 < \frac{\beta(2n+1)}{\sqrt{2n+1}} \leq 1 \quad (16)$$

The equality is true if and only if  $2n+1$  is a square.

The Dirichlet series :

$$\zeta_\beta(s + \frac{1}{2}) = \sum_{n=0}^{\infty} \frac{\beta(n)}{n^{s+\frac{1}{2}}} = \sum_{n=1}^{\infty} \frac{\beta(n)}{\sqrt{n}} \frac{1}{n^s} \quad (17)$$

is, following (13) convergent for  $\Re(s) > 1/2$ .

The following inequality is useful :

$$\sum_{n=1}^{\infty} \frac{|\beta(n)|}{n^{3/2}} \leq \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \sum_{h=1}^{\infty} \frac{1}{h^2} \quad (18)$$

## 2.4 The function $\zeta_\nu$ .

Let  $\zeta_\nu$  be :

$$\zeta_\nu(s) = \frac{1}{\zeta_{imp}(s+1)} \frac{\zeta_{imp}(2s+2)}{\zeta_{imp}(s+3/2)} = \frac{\zeta_\beta(s+3/2)}{\zeta_{imp}(s+1)} \quad (19)$$

It is the generating function of an arithmetic function  $\nu$  :

$$\zeta_\nu(s) = \sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} \quad \Re(s) > 0 \quad (20)$$

Of course,  $\nu$  is zero on even integers:  $\nu(2m) = 0$ .

$$\sum_{l|(2n+1)} l\nu(l) = \frac{\beta(2n+1)}{\sqrt{2n+1}} \quad (21)$$

the Möbius formula gives :

$$(2n+1)\nu(2n+1) = \sum_{kl=2n+1} \mu(k) \frac{\beta(l)}{\sqrt{l}} \quad (22)$$

Let  $d$  be the arithmetic function  $d(n)$  = number of divisors of  $n$ . We have an estimate for all integers  $m$  :

$$|\nu(m)| \leq \frac{d(m)}{m} \quad (23)$$

**Theorem 1.** *The series whose terms are  $\nu(n)$  converge and the value is 0 .*

$$\sum_{n=1}^{\infty} \nu(n) = 0. \quad (24)$$

Proof. Use (22), (18) and (14).  $\square$

This result is very important for the present work .

### 3 Integral formula for the $\zeta_a$ function

#### 3.1 The kernel $\frac{1}{e^z+1}$ .

The kernel  $\frac{1}{e^z+1}$  is better than  $\frac{1}{e^z-1}$  because we do not have a singularity at the origin. We have some classical expansions :

$$\begin{aligned} \frac{1}{e^t+1} &= \sum_{n=1}^{\infty} (-1)^{n-1} e^{-nt} \quad (t > 0) \\ \frac{1}{e^z+1} &= \frac{1}{2} - 2z \sum_{n=0}^{\infty} \frac{1}{z^2 + (2n+1)^2 \pi^2} \end{aligned} \quad (25)$$

Now, we take  $|z| < \pi$  for convergence .

$$\frac{1}{e^z+1} = \frac{1}{2} - 2z \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{((2n+1)\pi)^{2k}} \quad (26)$$

$$\frac{1}{e^z+1} = \frac{1}{2} - 2 \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\pi^{2k+2}} \zeta_{imp}(2k+2) \quad (27)$$

#### 3.2 Integral representation formula for $\zeta_a$ .

It is well known that :

$$\Gamma(s) \zeta_a(s) = \int_0^{\infty} \frac{1}{e^t+1} t^{s-1} dt \quad \Re(s) > 0 \quad (28)$$

The continuation of this integral representation is possible by taking :

$$\frac{1}{e^t+1} - \frac{1}{2}$$

### 4 Functional equations

#### 4.1 Functional equation between $\zeta_a$ and $\zeta_{imp}$ .

For  $\zeta$ , the following functional equation of Riemann is well known cf [6] :

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi}{2}s\right) \Gamma(1-s) \zeta(1-s) \quad (29)$$

thence a functional equation between  $\zeta_a$  and  $\zeta_{imp}$  :

$$\zeta_a(s) = -2\pi^{s-1} \sin\left(\frac{\pi}{2}s\right) \Gamma(1-s) \zeta_{imp}(1-s) \quad (30)$$

## 4.2 Functional equation between $\zeta_\alpha$ and $\zeta_\beta$ .

From (30) :

$$\frac{\zeta_a(2s)}{\zeta_a(s)} = \frac{-2\pi^{2s-1} \sin(\pi s) \Gamma(1-2s) \zeta_{imp}(1-2s)}{-2\pi^{s-1} \sin(\frac{\pi}{2}s) \Gamma(1-s) \zeta_{imp}(1-s)}$$

Hence (cf [2] for the duplication formula) a first form for the functional equation between  $\zeta_\alpha$  and  $\zeta_\beta$  :

$$\zeta_\alpha(s) = 2^{1-2s} \pi^{s-1/2} \cos\left(\frac{\pi}{2}s\right) \Gamma\left(\frac{1}{2}-s\right) \zeta_\beta(1-s) \quad (31)$$

And a second form, but only for  $\Re(s) < 0$  :

$$\zeta_\alpha(s) = 2^{1-2s} \cos\left(\frac{\pi}{2}s\right) \Gamma\left(\frac{1}{2}-s\right) \sum_{m=0}^{\infty} \frac{\beta(2m+1)}{\sqrt{2m+1}} \frac{1}{(\pi(2m+1))^{1/2-s}} \quad (32)$$

## 5 Integral formulas

### 5.1 Integral formula for $\zeta_\alpha$ in the domain $-3/2 < \Re(s) < -1/2$ .

The functional equation (32) gives an integral for  $\zeta_\alpha$  in the domain  $-3/2 < \Re(s) < 0$  :

$$\zeta_\alpha(s) = 2^{1-2s} \cos\left(\frac{\pi}{2}s\right) \Gamma\left(\frac{1}{2}-s\right) \frac{2}{\pi} \cos\left(\frac{\pi}{2}s + \frac{\pi}{4}\right) \sum_{m=0}^{\infty} \frac{\beta(2m+1)}{\sqrt{2m+1}} \int_0^{\infty} \frac{x^{s+1/2}}{x^2 + \pi^2(2m+1)^2} dx$$

Now, we want to permute the summation and the integral. To do this we have only to prove absolute integrability, but for  $-3/2 < \Re(s) < -1/2$  . Let  $\sigma = \Re(s)$ , take the inequality (16) :

$$\int_0^{\infty} 2x \sum_{n=1}^N \left| \frac{\beta(n)}{\sqrt{n}(x^2 + \pi^2 n^2)} \right| |x^{s-1/2}| dx \leq \int_0^{\infty} \left( \frac{1}{2} - \frac{1}{e^x + 1} \right) x^{\sigma-1/2} dx \quad -3/2 < \Re(s) < -1/2$$

We get an integral formula for  $\zeta_\alpha$  in  $-3/2 < \Re(s) < -1/2$  :

$$\zeta_\alpha(s) = \frac{2^{1-2s}}{\pi} \cos\left(\frac{\pi}{2}s\right) \cos\left(\frac{\pi}{2}s + \frac{\pi}{4}\right) \Gamma(1/2-s) \int_0^{\infty} 2x \sum_{m=0}^{\infty} \frac{\beta(2m+1)}{\sqrt{2m+1}(x^2 + \pi^2(2m+1)^2)} x^{s-1/2} dx \quad (33)$$

Let :

$$\begin{aligned} \varphi(s) &= \frac{2^{1-2s}}{\pi} \cos\left(\frac{\pi}{2}s\right) \cos\left(\frac{\pi}{2}s + \frac{\pi}{4}\right) \Gamma(1/2-s) \\ \mathcal{N}(x) &= 2x \sum_{m=0}^{\infty} \frac{\beta(2m+1)}{\sqrt{2m+1}(x^2 + \pi^2(2m+1)^2)} \end{aligned}$$

Write (33) as :

$$\zeta_\alpha(s) = \varphi(s) \int_0^{\infty} \mathcal{N}(x) x^{s-1/2} dx \quad -3/2 < \Re(s) < -1/2 \quad (34)$$

Now, the aim is to prove this formula for a greater domain .

## 5.2 The meromorphic function $\mathcal{N}$ .

$$\mathcal{N}(z) = 2z \sum_{m=0}^{\infty} \frac{\beta(2m+1)}{\sqrt{2m+1}(z^2 + \pi^2(2m+1)^2)} \quad (35)$$

is a meromorphic function in  $\mathbb{C}$  .

All the poles are simple at  $i\pi(2m+1)$  for  $m \in \mathbb{Z}$ . The residu is :

$$\frac{\beta(2m+1)}{\sqrt{2m+1}}$$

The expansion of  $\mathcal{N}$  in a power series, in a neighborhood of zero is :

$$\mathcal{N}(z) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\pi^{2k+2}} \zeta_{\beta}(2k+5/2) \quad |z| < \pi \quad (36)$$

## 5.3 Definition of the meromorphic function $\mathcal{M}$ .

Let  $\mathcal{M}$  be the following meromorphic function in  $\mathbb{C}$  .

$$\mathcal{M}(z) = \sum_{m=0}^{\infty} \nu(2m+1) \left( \frac{1}{2} - \frac{1}{e^{z/(2m+1)} + 1} \right) \quad (37)$$

All the poles are simple at  $i(2l+1)\pi$  for  $l \in \mathbb{Z}$ . The residu of  $\mathcal{M}$  is :

$$\sum_{(2m+1)|(2l+1)} (2m+1)\nu(2m+1) = \frac{\beta(2l+1)}{\sqrt{2l+1}}$$

because of (21). We obtain the same poles and the same residus. Of course, this does not give the equality between  $\mathcal{N}$  and  $\mathcal{M}$ .

Theorem 1, (24) and the previous definition (37) of  $\mathcal{M}$  give us :

$$\mathcal{M}(z) = - \sum_{m=0}^{\infty} \nu(2m+1) \frac{1}{e^{z/(2m+1)} + 1} \quad (38)$$

## 5.4 Behavior of $\mathcal{M}$ at infinity.

By Abel's summation by parts on (37), and theorem 1, (24), we get :

$$\lim_{x \rightarrow \infty} \mathcal{M}(x) = 0 \quad (39)$$

## 5.5 A bound for the derivative $\mathcal{M}'$ .

We can derive term by term the series of  $\mathcal{M}(z)$  . From (38), we get :

$$\mathcal{M}'(z) = \sum_{m=0}^{\infty} \frac{\nu(2m+1)}{2m+1} \frac{e^{z/(2m+1)}}{(e^{z/(2m+1)} + 1)^2} \quad (40)$$

Now, take this for  $x$  real positive . There exists a constant  $C$  such that for all  $x \in [0, +\infty[$  :

$$|\mathcal{M}'(x)| \leq C \quad (41)$$

## 5.6 A better bound for $\mathcal{M}$ at infinity ?

Is it true that there exists a constant  $C$  such that :

$$|\mathcal{M}(x)| \leq \frac{1}{x} C \quad ? \quad (42)$$



## 5.7 Identity between $\mathcal{M}$ and $\mathcal{N}$ .

The purpose is to prove that in a neighborhood of 0, we have :

$$\mathcal{M}(z) = \mathcal{N}(z)$$

Starting from (27) we get for  $|z| < \pi$  :

$$\sum_{n=0}^{\infty} \nu(2n+1) \left( \frac{1}{2} - \frac{1}{e^{z/(2n+1)} + 1} \right) = 2 \sum_{n=0}^{\infty} \nu(2n+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{\pi^{2k+2}} \zeta_{imp}(2k+2) \frac{z^{2k+1}}{(2n+1)^{2k+1}}$$

We can switch the summations because we have absolute convergence. Hence :

$$\sum_{n=0}^{\infty} \nu(2n+1) \left( \frac{1}{2} - \frac{1}{e^{z/(2n+1)} + 1} \right) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\pi^{2k+2}} \zeta_{imp}(2k+2) \zeta_{\nu}(2k+1)$$

With (19), we get the power series of  $\mathcal{M}$  at the origin :

$$\mathcal{M}(z) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\pi^{2k+1}} \zeta_{\beta}(2k+5/2) \quad |z| < \pi$$

This is exactly (36) and we get that  $\mathcal{M}$  and  $\mathcal{N}$  are two expressions of the same function .

## 5.8 Integral representation formulas for $\zeta_{\lambda}$ in $-3/2 < \Re(s) < -1/2$

From (34) :

$$\zeta_{\alpha}(s) = \varphi(s) \int_0^{\infty} \mathcal{N}(x) x^{s-1/2} dx \quad (-3/2 < \Re(s) < -1/2)$$

and with (42) we have :

$$|\mathcal{N}(x)| = |\mathcal{M}(x)| \leq C/x$$

Hence the convergence of the integral at  $\infty$  for  $\Re(s) < 1/2$  .

**Theorem 2.** *In the domain  $-3/2 < \Re(s) < -1/2$ , we have the integral representation formula :*

$$\zeta_{\lambda}(s) = \frac{1 - 2^{1-s}}{1 - 2^{1-2s}} \varphi(s) \int_0^{\infty} \mathcal{N}(x) x^{s-1/2} dx \quad (43)$$

More explicitly, with the values of  $\varphi$ ,  $\mathcal{N}$  and  $\mathcal{M}$  :

$$\zeta_{\lambda}(s) = \frac{1 - 2^{1-s}}{(2^{2s-1} - 1)\pi} \cos\left(\frac{\pi}{2}s\right) \cos\left(\frac{\pi}{2}s + \frac{\pi}{4}\right) \Gamma(1/2 - s) \int_0^{\infty} \sum_{m=0}^{\infty} \frac{2x \beta(2m+1)}{\sqrt{2m+1}(x^2 + \pi^2(2m+1)^2)} x^{s-1/2} dx \quad (44)$$

$$\zeta_{\lambda}(s) = \frac{2^{1-s} - 1}{(2^{2s-1} - 1)\pi} \cos\left(\frac{\pi}{2}s\right) \cos\left(\frac{\pi}{2}s + \frac{\pi}{4}\right) \Gamma(1/2 - s) \int_0^{\infty} \sum_{m=0}^{\infty} \frac{\nu(2m+1)}{e^{x/(2m+1)} + 1} x^{s-1/2} dx \quad (45)$$

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### **Acknowledgements**

I thank Michel Paugam for his many fruitful remarks and for his trust in my work, and also Claude Longuemare for what he taught to me through different subjects we studied.